Chapter 4

The Lebesgue Integral

4.1 Measurable functions

Definition 4.1.1. Let $\mathbb{X}$ and $\mathbb{Y}$ be arbitrary sets and let $\mathfrak{S}_{\mathbb{X}}$ and $\mathfrak{S}_{\mathbb{Y}}$ be the systems of their subsets. An abstract function $f : \mathbb{X} \to \mathbb{Y}$ is called $(\mathfrak{S}_{\mathbb{X}}, \mathfrak{S}_{\mathbb{Y}})$-measurable if $A \in \mathfrak{S}_{\mathbb{Y}}$ implies that $f^{-1}(A) \in \mathfrak{S}_{\mathbb{X}}$.

Note that although one says "measurable", there is no measures in the Definition 4.1.1.
For instance, if we let $\mathbb{X} := \mathbb{R}^1$, $\mathbb{Y} := \mathbb{R}^1$ and $\mathfrak{S}_{\mathbb{X}}$, $\mathfrak{S}_{\mathbb{Y}}$ be the systems of all open sets then a function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is measurable iff it is continuous.
But one usually can assign measures on $\mathfrak{S}_{\mathbb{X}}$ and $\mathfrak{S}_{\mathbb{Y}}$ and, as you guess, the most interesting case is when $\mathfrak{S}_{\mathbb{X}}$ and $\mathfrak{S}_{\mathbb{Y}}$ are the σ-algebras.

The idea of measurable function is critical for the theory of the Lebesgue integral as well as for probabability theory.

Definition 4.1.2. Consider a measure space, i.e. a triple $(\mathbb{X}, \mathcal{A}, \mu)$, where $\mathbb{X}$ is an abstract space, $\mathcal{A}$ is a σ-algebra of subsets of $\mathbb{X}$ and μ is a measure on $\mathcal{A}$.
A real function $f : \mathbb{X} \to \mathbb{R}$ is called μ-measurable if for any Borel(!) set $A \subset \mathbb{R}$ holds that $f^{-1}(A) \in \mathcal{A}$

Remark 4.1.3. if $\mathbb{X}$ if a probability space then f is a [real valued] random variable, its μ-measurability means that for every Borel set on $\mathbb{R}$ there is an event in $\mathcal{A}$. The converse is not necessarily true, i.e. there may be an event $E \in \mathcal{A}$ such that $f(E) \subset \mathbb{R}$ is not a Borel set.

In what follows I will usually write "measurable" instead of "μ-measurable" unless it causes a confusion.

Using only the ideas from Real Analysis it is hard (though possible) to construct an non-measurable function. But if we engage very basic probability theory then here is a simple and very edifying example. Let us toss a coin two times. If we obtain head twice I pick 2$ otherwise I get nothing. If, in turn, there is a head at the 1st toss, you pick 1$ (the second toss is irrelevant for you). Denote our probability space as Ω. The powerset of Ω is the σ-algebra, which contains all events, i.e.

$$\mathcal{P}(\Omega) = \{\Omega, \varnothing, HT, HH, TH, TT, \{HH \cup HT\}, \{TH \cup TT\}\}$$

So my profit is a mapping $G : \Omega \to R$ and yours is $\bar{G} : \Omega \to R$. Both mappings are measurable w.r.t. $\mathcal{P}(\Omega)$. However, if we are given only an information about the 1st toss, we can describe it as the following σ-algebra

$$\mathcal{A} := \{\Omega, \varnothing, \{HH \cup HT\}, \{TH \cup TT\}\}$$

$\bar{G}$ is measurable w.r.t. $(\Omega, \mathcal{A})$ but G is not. Indeed, consider an event that my profit is more than 1$. The set $G^{-1}((1, \infty)) = \{\omega : G > 1\} = HH$ is not in $\mathcal{A}$, whereas $(1, \infty)$ is obviously a Borel set.
Note, if we e.g. consider an event that I got more than 2$ then $G^{-1}((2, \infty)) = \{\omega : G > 2\} = \varnothing$ is in $\mathcal{A}$. But as you remember, according to the Definition 4.1.1 any(!) Borel set must have a counterpart in $\mathcal{A}$.